OPERATIONS WITH RESPECT TO WHICH THE ELEMENTS OF A BOOLEAN ALGEBRA FORM A GROUP*

BY

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In a previous paper† I pointed out the existence of two operations with respect to each of which the elements of a boolean algebra form an abelian group. If we denote the logical sum of two elements a, b by $a + b$, their logical product by ab, and the negative of an element a by a', then the two operations in question are given by $ab' + a'b$, $ab + a'b'$. In the present paper I determine *all* the operations with respect to which the elements of a boolean algebra form a group in general and an abelian group in particular.

Postulates for groups.‡ A class K of elements $a, b, c, \ldots$ is a *group* with respect to an operation $\bigcirc$ if the following two conditions are satisfied:

P_1. $(a \bigcirc b) \bigcirc c = a \bigcirc (b \bigcirc c)$,

whenever $a, b, c, a \bigcirc b, b \bigcirc c, a \bigcirc (b \bigcirc c)$ are elements of K.

P_2. For any two elements a, b, in K there exists an element x such that $a \bigcirc x = b$.

The group is *abelian* if the following condition also is satisfied:

P_3. $a \bigcirc b = b \bigcirc a$,

whenever $a, b, b \bigcirc a$ are elements of K.

Determination of group operations. We shall have all the operations of a boolean algebra with respect to which the elements form a group if we determine for groups in general all the boolean operations which have the properties P_1, P_2, and for abelian groups, all the operations which have the properties P_1, P_2, P_3. I proceed to effect this determination.

If $f(x, y)$ is any determinate function of two elements x, y of a boolean algebra, then

$$f(x, y) = f(1, 1)xy + f(1, 0)xy' + f(0, 1)x'y + f(0, 0)x'y',$$

where 1 and 0 are respectively the *whole* and the *zero* of the algebra. Hence, any class-closing operation $\bigcirc$ on two boolean elements a, b is given by

$$a \bigcirc b = Aab + Bab' + Ca'b + Da'b', \tag{1}$$

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† *Complete sets of representations of two-element algebras,* Bulletin of the American Mathematical Society, vol. 30, pp. 24–30.

‡ See these Transactions, vol. 4 (1903), p. 27.

where the *discriminants* A, B, C, D, which determine the operation $\circ$, are elements of the algebra. All operations $\circ$ with respect to which the elements of a boolean algebra form a group are then given by the discriminants A, B, C, D which will make operation (1) satisfy postulates P_1, P_2 in case of the general group, and postulates P_1, P_2, P_3 in case of the abelian.

Now from (1)

$$
\begin{aligned}
(a\circ b)\circ c &= (Aab + Bab' + Ca'b + Da'b')\circ c\\
&= A(Aabc + Bab'c + Ca'bc + Da'b'c)\\
&\quad + B(Aabc' + Bab'c' + Ca'bc' + Da'b'c')\\
\text{(i)}\qquad &\quad + C(A'abc + B'ab'c + C'a'bc + D'a'b'c)\\
&\quad + D(A'abc' + B'ab'c' + C'a'bc' + D'a'b'c')\\
&= (A + C)abc + (BA + DA')abc' + (AB + CB')ab'c\\
&\qquad\qquad + (B + D)ab'c'\\
&\quad + ACa'bc + (BC + DC')a'bc' + (AD + CD')a'b'c\\
&\qquad\qquad + BDa'b'c';
\end{aligned}
$$

and

$$
\begin{aligned}
a\circ(b\circ c) &= a\circ(Abc + Bbc' + Cb'c + Db'c')\\
&= A(Aabc + Babc' + Cab'c + Dab'c')\\
&\quad + B(A'abc + B'abc' + C'ab'c + D'ab'c')\\
\text{(ii)}\qquad &\quad + C(Aa'bc + Ba'bc' + Ca'b'c + Da'b'c')\\
&\quad + D(A'a'bc + B'a'bc' + C'a'b'c + D'a'b'c')\\
&= (A + B)abc + ABabc' + (AC + BC')ab'c + (AD + BD')ab'c'\\
&\quad + (CA + DA')a'bc + (CB + DB')a'bc' + (C + D)a'b'c\\
&\quad + CDa'b'c'.
\end{aligned}
$$

Using postulate P_1, and equating corresponding discriminants of (i) and (ii), we get

$$
\begin{gathered}
A + C = A + B, \quad BA + DA' = AB, \quad AB + CB' = AC + BC',\\
B + D = AD + BD', \quad AC = CA + DA', \quad BC + DC' = CB + DB',\\
AD + CD' = C + D, \quad BD = CD;
\end{gathered}
$$

or

$$A'B'C + A'BC' + A'D + BC'D + B'CD = 0,$$

or

(2) $$D = AD, \quad (BC' + B'C)(AD + A'D') = 0.$$

The condition that the operation $\circ$ given by (1) satisfy postulate P_2 is the condition that for two given elements a, b there be a solution for x of the equation

$$Aax + Bax' + Ca'x + Da'x' = b,$$

or of the equation

(iii) $$(A'ab + Aab' + C'a'b + Ca'b')x + (B'ab + Bab' + D'a'b + Da'b')x' = 0.$$

The condition that (iii) have a solution is

$$(A'ab + Aab' + C'a'b + Ca'b')(B'ab + Bab' + D'a'b + Da'b') = 0,$$

or

(iv) $$A'B'ab + ABab' + C'D'a'b + CDa'b' = 0.$$

The conditions that (iv) hold for *any* elements a, b, are

$$A'B' = 0, \quad AB = 0, \quad C'D' = 0, \quad CD = 0,$$

which reduce to

(3) $$B = A', \qquad C = D'.$$

Finally, the condition that the operation $\circ$ of (1) satisfy postulate P_3 is that (1) be symmetric in a, b. The condition for this is

(4) $$B = C.$$

Conditions (2), (3), (4) are sufficient as well as necessary in order that operation (1) satisfy postulates P_1, P_2, P_3 respectively.

From (2) and (3), the conditions that the operation (1) satisfy P_1, P_2 simultaneously are

$$B = A', \quad C = D', \quad D = AD, \quad (BC' + B'C)(AD + A'D') = 0,$$

which conditions reduce to

(5) $$B = A', \qquad C = D', \qquad D = AD.$$

Hence

THEOREM 1. *The totality of operations with respect to which the elements of a boolean algebra form a group is given by*

$$Aab + A'ab' + D'a'b + Da'b', \quad D = AD. \tag{6}$$

From (4) and (5), the conditions that operation (1) satisfy postulates P_1, P_2, P_3 simultaneously are

$$B = A', \quad C = D', \quad D = AD, \quad C = B,$$

which reduce to

$$B = A', \quad C = A', \quad D = A. \tag{7}$$

Hence

THEOREM 2. *The totality of operations with respect to which the elements of a boolean algebra form an abelian group is given by*

$$Aab + A'ab' + A'a'b + Aa'b'. \tag{8}$$

Remarks. 1. For the general group, the element x demanded by postulate P_2 is, from (iii) and (5),

$$x = Aab + A'ab' + D'a'b + Da'b', \quad D = AD. \tag{9}$$

For abelian groups, from (iii) and (7),

$$x = Aab + A'ab' + A'a'b + Aa'b'. \tag{10}$$

2. From (2), *the totality of boolean operations which obey the associative law is given by*

$$Aab + Bab' + Ca'b + Da'b', \quad D = DA, \quad (BC' + B'C)(AD + A'D') = 0. \tag{11}$$

3. From (3), *the totality of binary boolean operations which always have an inverse is given by*

$$Aab + A'ab' + D'a'b + Da'b'. \tag{12}$$

4. From (4), *the totality of boolean operations which obey the commutative law is given by*

$$Aab + Bab' + Ba'b + Da'b'. \tag{13}$$

5. From (2) and (4), *the totality of boolean operations which are both associative and commutative is given by*

$$\begin{gathered} Aab + Bab' + Ba'b + Da'b', \\ D = AD. \end{gathered} \tag{14}$$

6. From (2) and (3), *the totality of associative boolean operations which always have an inverse is given by*

$$\begin{gathered} Aab + A'ab' + D'a'b + Da'b', \\ D = AD. \end{gathered} \tag{15}$$

7. From (3) and (4), *the totality of commutative boolean operations which always have an inverse is given by*

$$Aab + A'ab' + A'a'b + Aa'b'. \tag{16}$$

8. Since (16) is the same as (8), *a commutative boolean operation which always has an inverse is also associative, and is an abelian group operation.*

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